Stability and Relevance in Incomplete Argumentation Frameworks

Daphne ODEKERKEN [a,b,1], AnneMarie BORG [a] and Floris BEX [a,c]
[a] *Department of Information and Computing Sciences, Utrecht University*
[b] *National Police Lab AI, Netherlands Police*
[c] *Tilburg Institute for Law, Technology and Society, Tilburg University*
ORCiD ID: Daphne Odekerken https://orcid.org/0000-0003-0285-0706, AnneMarie Borg https://orcid.org/0000-0002-7204-6046, Floris Bex https://orcid.org/0000-0002-5699-9656

Abstract. We explore the computational complexity of stability and relevance in incomplete argumentation frameworks (IAFs), abstract argumentation frameworks that encode qualitative uncertainty by distinguishing between certain and uncertain arguments and attacks. IAFs can be specified by, e.g., making uncertain arguments or attacks certain; the justification status of arguments in an IAF is determined on the basis of the certain arguments and attacks. An argument is *stable* if its justification status is the same in all specifications of the IAF. For arguments that are not stable in an IAF, the *relevance* problem is of interest: which uncertain arguments or attacks should be investigated for the argument to become stable? We redefine stability and define relevance for IAFs and study their complexity.

Keywords. Incomplete argumentation frameworks, stability, relevance, complexity

1. Introduction

Computational argumentation is an important research field in artificial intelligence, concerning reasoning with incomplete or inconsistent information [1]. A central concept are argumentation frameworks (AFs): a set of arguments and an attack relation between them [2]. Given an AF and so-called semantics, one can determine extensions: sets of arguments that can collectively be considered to be acceptable. Based on these extensions, each argument has at least one justification status (in terms of e.g. labels like IN, OUT and UNDEC). However, in practice, argumentation is a dynamic process in which not all arguments may be known in advance. Incomplete argumentation frameworks (IAFs) are designed to handle this dynamic process by extending AFs to allow for both certain and uncertain arguments and attacks [3,4,5,6]. In this paper, we study two problems in IAFs and their complexity for various semantics: stability and relevance.

Detecting stability was initially introduced for the ASPIC$^+$ framework in [7] and subsequently studied for structured and abstract argumentation settings in [8,6]. Informally, an argument is stable if more information cannot change its justification status.

[1] Corresponding Author; E-mail: d.odekerken@uu.nl

Stability detection has practical applications, for instance as a termination criterion for argumentative dialogue agents: in the agent architecture for inquiry proposed in [8], stability detection prevents the agent from asking unnecessary questions. In addition, [6] proposes an application of stability detection in negotiating agents, to recognise situations in which an agent should stop negotiating and accept its opponent's offer.

For situations in which the argument of interest is not stable in the given IAF, a natural question would be: which uncertainties should we resolve in order to reach a point where the argument is stable? In other words: which uncertain arguments or attacks are still relevant for the justification status? Adding relevance to an inquiry/negotiation process ensures that the questions that are asked contribute to reaching stability.

The contribution of this paper is the extensive study of both stability and relevance in the context of IAFs. Specifically, first we (re)define stability on IAFs, considering not only IN, but also OUT and UNDEC justification statuses. This results in a more fine-grained notion of stability than an earlier definition in [6]. Second, we present precise complexity results for stability of all these justification statuses in grounded, complete, stable and preferred semantics, refining preliminary results in [6]. Third, we define a notion of relevance in the context of reaching stability in IAFs. Finally, we present preliminary complexity results for the introduced relevance detection problem.

The paper is structured as follows. In Section 2, we provide the necessary preliminaries. In Section 3, we study the complexity of identifying the justification status of an argument and use these results in our complexity analysis of the stability problem. We then introduce the relevance problem for IAFs in Section 4 and provide complexity results. Related work is discussed in Section 5; we conclude in Section 6.

2. Preliminaries

In this section, we recall the most important notions from abstract argumentation and the considered semantics [2] as well as incomplete argumentation frameworks [3,4,5,6] and their specifications. Finally, we give a brief introduction to the polynomial hierarchy, which is required for our complexity study.

2.1. Argumentation frameworks and semantics

An argumentation framework $\langle \mathcal{A}, \mathcal{R} \rangle$ (AF) consists of a finite set $\mathcal{A}$ of arguments and a binary attack relation $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ on them, where $(A,B) \in \mathcal{R}$ indicates that argument A attacks argument B. The evaluation of arguments is done using the semantics of [2].

Definition 1 (Extension-based semantics)**.** *Let $AF = \langle \mathcal{A}, \mathcal{R} \rangle$ be an AF and $S \subseteq \mathcal{A}$. Then:*

- *S is **conflict-free** iff for each $X, Y \in S : (X,Y) \notin \mathcal{R}$;*
- *$X \in \mathcal{A}$ is **acceptable with respect to** S iff for each $Y \in \mathcal{A}$ such that $(Y,X) \in \mathcal{R}$, there is a $Z \in S$ such that $(Z,Y) \in \mathcal{R}$;*
- *S is an **admissible set** iff S is conflict free and $X \in S$ implies that X is acceptable with respect to S;*
- *S is a **complete extension** (CP) iff S is admissible and for each X: if $X \in \mathcal{A}$ is acceptable with respect to S then $X \in S$;*
- *S is a **preferred extension** (PR) iff it is the set inclusion maximal admissible set;*

- *S is the **grounded extension** (GR) iff it is the set inclusion minimal complete extension; and*
- *S is a **stable extension** (ST) iff it is complete and attacks all the arguments in $\mathcal{A} \setminus S$.*

2.2. Incomplete argumentation frameworks

Incomplete argumentation frameworks (IAFs) are an extension to AFs, initially proposed as partial AFs in [3] and later studied as IAFs in e.g. [4,5,6]. In an IAF, the set of arguments and attacks is split into two disjoint parts: a certain part ($\mathcal{A}$ and $\mathcal{R}$) and an uncertain part ($\mathcal{A}^?$ and $\mathcal{R}^?$). For the uncertain elements, it is not known whether they are part of the argumentation framework or not. They may be added in the future, for example because more information is acquired in an inquiry dialogue, or removed, for example because after investigation, this element turned out not to be present in the given setting.

Definition 2 (Incomplete argumentation framework)**.** *An incomplete argumentation framework is a tuple $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, where $\mathcal{A} \cap \mathcal{A}^? = \emptyset$, $\mathcal{R} \cap \mathcal{R}^? = \emptyset$ and:*

- *$\mathcal{A}$ is the set of certain arguments;*
- *$\mathcal{A}^?$ is the set of uncertain arguments;*
- *$\mathcal{R} \subseteq (\mathcal{A} \cup \mathcal{A}^?) \times (\mathcal{A} \cup \mathcal{A}^?)$ is the certain attack relation; and*
- *$\mathcal{R}^? \subseteq (\mathcal{A} \cup \mathcal{A}^?) \times (\mathcal{A} \cup \mathcal{A}^?)$ is the uncertain attack relation.*

An IAF can be *specified* by obtaining more information about the uncertain part.

Definition 3 (Specification)**.** *Given an IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a specification is an IAF $\mathcal{I}' = \langle \mathcal{A}', \mathcal{A}^{?\prime}, \mathcal{R}', \mathcal{R}^{?\prime} \rangle$, where:*

- *$\mathcal{A} \subseteq \mathcal{A}' \subseteq \mathcal{A} \cup \mathcal{A}^?$;*
- *$\mathcal{R} \subseteq \mathcal{R}' \subseteq \mathcal{R} \cup \mathcal{R}^?$;*
- *$\mathcal{A}^{?\prime} \subseteq \mathcal{A}^?$;*
- *$\mathcal{R}^{?\prime} \subseteq \mathcal{R}^?$.*

We denote all possible specifications for $\mathcal{I}$ by $F(\mathcal{I})$. Note that $\mathcal{A}' \cap \mathcal{A}^{?\prime} = \emptyset$ and $\mathcal{R}' \cap \mathcal{R}^{?\prime} = \emptyset$ because $\mathcal{I}'$ is an IAF.

Since the semantics of [2] are defined on AFs, we define the certain projection of an IAF, which is an AF consisting of only the IAF's certain arguments and attacks.

Definition 4 (Certain projection)**.** *Given an IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, the certain projection is the argumentation framework $AF = \langle \mathcal{A}, \mathcal{R} \cap (\mathcal{A} \times \mathcal{A}) \rangle$.*

Note that our definition of specification is similar to the notion of completion used in related work on IAFs [4,5]. Intuitively, a completion is a certain projection of a specification and is therefore not suitable for keeping track of the uncertain elements. Since the development of uncertain elements is essential for defining and studying the relevance problem (see Section 4), we use the notion of specification rather than completion.

2.3. The polynomial hierarchy

The polynomial hierarchy [9] is a hierarchy of complexity classes above NP defined using oracle machines, i.e. Turing machines that are allowed to call a subroutine (oracle), deciding some fixed problem in constant time. For a class of decision problems $\mathcal{C}$ and a class $\mathcal{X}$ defined by resource bounds, $\mathcal{X}^{\mathcal{C}}$ denotes the class of problems decidable on a Turing machine with a resource bound given by $\mathcal{X}$ and an oracle for a problem in $\mathcal{C}$.

Based on these notions, the sets Σ_k^p and Π_k^p are defined as follows: $\Sigma_0^p = \Pi_0^p = \Delta_0^p = P$, $\Sigma_{k+1}^p = \mathrm{NP}^{\Sigma_k^p}$ and $\Pi_{k+1}^p = \mathrm{CoNP}^{\Sigma_k^p}$. The canonical complete problem for Σ_k^p is k-QBF, which is the problem of deciding whether the quantified boolean formula with k alternating quantifiers, starting with an existential quantifier, is true for a formula Φ – for example, deciding if $\exists X \text{s.t.} \ \forall Y : \Phi[X,Y] = \text{True}$ is Σ_2^p-complete. The complement of a k-QBF problem, denoted by co-k-QBF, is complete for Π_k^p.

3. Justification status and stability

In order to define and study stability and relevance, we need a definition of justification status. Given an AF $\langle \mathcal{A}, \mathcal{R} \rangle$, an argument A and a semantics σ, A's justification status can be determined by either considering all σ-extensions (sceptical) or at least one σ-extension of the AF (credulous). In this context, an argument can be IN (part of all/some σ-extensions); OUT (attacked by all/some σ-extensions), or UNDEC (otherwise) [10].

Definition 5 (Argument justification status)**.** *Let $AF = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework and σ some semantics in $\{\text{GR}, \text{CP}, \text{PR}, \text{ST}\}$. Let A be some argument in $\mathcal{A}$.*

- *A is σ-sceptical-IN (resp. σ-credulous-IN) iff A belongs to each (resp. some) σ-extension of AF;*
- *A is σ-sceptical-OUT (resp. σ-credulous-OUT) iff for each (resp. some) σ-extension S of AF, A is attacked by some argument in S;*
- *A is σ-sceptical-UNDEC (resp. σ-credulous-UNDEC) iff for each (resp. some) σ-extension of AF, A is not in S and not attacked by any argument in S.*

The justification statuses that we consider in this paper are $\{\text{GR}, \text{CP}, \text{PR}, \text{ST}\} \times \{\text{sceptical}, \text{credulous}\} \times \{\text{IN}, \text{OUT}, \text{UNDEC}\}$. Based on these justification statuses, we can now define stability:

Definition 6 (Stability on IAFs)**.** *Given an IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a certain argument $A \in \mathcal{A}$ and some justification status j, A is* stable-j *w.r.t. $\mathcal{I}$ iff for each specification $\mathcal{I}' = \langle \mathcal{A}', \mathcal{A}^{?\prime}, \mathcal{R}', \mathcal{R}^{?\prime} \rangle$ in $F(\mathcal{I})$, A is j w.r.t. the certain projection of $\mathcal{I}'$.*

Note that whereas the sceptical stability variants are mutually exclusive, this does not apply for the credulous stability variants, where an argument may have multiple stability statuses at the same time. Consider the following example:

Example 1. *Let $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ be an incomplete argumentation framework with $\mathcal{A} = \{A, B\}$, $\mathcal{A}^? = \mathcal{R}^? = \emptyset$, $\mathcal{R} = \{(A,B),(B,A)\}$; note that $F(\mathcal{I}) = \{\mathcal{I}\}$. Both A and B are stable-CP-credulous-IN, stable-CP-credulous-OUT and stable-CP-credulous-UNDEC. For $\sigma \in \{\text{ST}, \text{PR}\}$, both A and B are stable-σ-credulous-IN and stable-σ-credulous-OUT. On the other hand, for semantics $\sigma \in \{\text{CP}, \text{ST}, \text{PR}\}$, A and B are not stable-σ-sceptical-IN, -OUT or -UNDEC.*

An alternative definition of stability on IAFs has been proposed before in [6], but this definition only takes the IN status of arguments into account. Since other justification statuses are not studied, it is not possible to distinguish for example situations in which the justification status of an argument is UNDEC in each certain projection from situations where the justification status is always either OUT or UNDEC. Our definition of stability on IAFs is more fine-grained as it also includes OUT- and UNDEC-stability. In addition, we will provide precise complexity results, refining preliminary complexity bounds from [6].

We formulate the identification of justification and stability statuses as decision problems:

j-JUSTIFICATION	
Given:	An argumentation framework $\langle \mathcal{A}, \mathcal{R} \rangle$ and an argument $A \in \mathcal{A}$
Question:	Does A's justification status in $\langle \mathcal{A}, \mathcal{R} \rangle$ equal j?

j-STABILITY	
Given:	An incomplete argumentation framework $\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a justification status j and an argument $A \in \mathcal{A}$
Question:	Does A's stability status w.r.t. $\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ equal stable-j?

For some variants of the stability problem, we can directly use complexity results from earlier work, in particular the results on necessary sceptical and credulous acceptance presented in [5]. Specifically, for a given semantics $\sigma \in \{\text{GR}, \text{CP}, \text{PR}, \text{ST}\}$, an argument is stable-$\sigma$-sceptical-IN iff it is necessary sceptically accepted w.r.t. the corresponding IAF; similarly, the set of stable-σ-credulous-OUT arguments coincides with the necessary credulously accepted arguments (see Section 5 for a discussion). See Table 1 for an overview of these results.

σ	c/s	status	JUSTIFICATION	STABILITY	RELEVANCE
ST	c	IN/OUT	NP-c [11,12]	Π_2^p-c [5]	
ST	c	UNDEC	**Trivial (no)**	**Trivial (no)**	
ST	s	IN/OUT	CoNP-c [11]	CoNP-c (Π_2^p-c) [5]	
ST	s	UNDEC	CoNP-c [11] (**Trivial (no)**)	**CoNP-c** (**Trivial (no)**)	
CP	c	IN/OUT	NP-c [11,12]	Π_2^p-c [5]	
CP	c	UNDEC	**P-c**	**CoNP-c**	
CP	s	IN/OUT	P-c [2]	CoNP-c [5]	**NP-c**
CP	s	UNDEC	**CoNP-c**	**CoNP-c**	
GR	c	IN/OUT	P-c [2]	CoNP-c [5]	**NP-c**
GR	c	UNDEC	**P-c**	**CoNP-c**	
GR	s	IN/OUT	P-c [2]	CoNP-c [5]	**NP-c**
GR	s	UNDEC	**P-c**	**CoNP-c**	
PR	c	IN/OUT	NP-c [11,12]	Π_2^p-c [5]	
PR	c	UNDEC	$\boldsymbol{\Sigma_2^p}$**-c**	$\boldsymbol{\Pi_3^p}$**-c**	
PR	s	IN/OUT	Π_2^p-c [13]	Π_2^p-c [5]	
PR	s	UNDEC	**CoNP-c**	**CoNP-c**	

Table 1. Overview of all complexity results related to this paper. If a reference is specified, this complexity result is trivial from an earlier result in the literature. New results are printed bold; their proofs can be found in the appendix. Results for ST-sceptical-existence justification and stability are given in parentheses.

The complexity of OUT-STABILITY has not been studied, but can be derived from the complexity of identifying IN-STABILITY by a reduction from the corresponding IN-JUSTIFICATION problems. In the following lemma, we show that these complexities are the same for each of the semantics considered in this paper:[2]

Lemma 1. *For any given $\sigma \in \{\text{GR}, \text{CP}, \text{PR}, \text{ST}\}$ and $c \in \{sceptical, credulous\}$, the complexity of σ-c-OUT-STABILITY equals the complexity of σ-c-IN-STABILITY.*

A similar result exists for IN-JUSTIFICATION and OUT-JUSTIFICATION. The results for UNDEC-JUSTIFICATION can be found in Table 1 and the appendix. The complexity of UNDEC-STABILITY cannot be derived in a general way from e.g. IN-STABILITY: the approach depends on the chosen semantics. We can however provide a general upper bound based on the complexity of UNDEC-JUSTIFICATION:

Proposition 1 (Upper bound j-STABILITY)**.** *Given an IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a certain argument $A \in \mathcal{A}$ and a justification status j, if the complexity of j-JUSTIFICATION in $\langle \mathcal{A}, \mathcal{R} \rangle$ is $\mathcal{C}$, the j-STABILITY problem given A and $\mathcal{I}$ is in CoNP$^{\mathcal{C}}$.*

Proof. In a negative instance $(\mathcal{I}, A)$ of j-STABILITY, there is some $\mathcal{I}' \in F(\mathcal{I})$ such that A is not j in the certain projection of $\mathcal{I}'$. A polynomial-size certificate for this instance would for example be a specification $\mathcal{I}' = \langle \mathcal{A}', \emptyset, \mathcal{R}', \emptyset \rangle$ in $F(\mathcal{I})$ such that A is **not** j in $AF' = \langle \mathcal{A}', \mathcal{R}' \cap (\mathcal{A}' \times \mathcal{A}') \rangle$. Verification that $\mathcal{I}' \in F(\mathcal{I})$ can be done in polynomial time; the j-justification status check of A in AF' is done by calling a $\mathcal{C}$ oracle. □

Note that the complexity of identifying the stability or justification status for UNDEC statuses may be higher or lower than the complexity of identifying the IN or OUT status. For example, for ST semantics the credulous JUSTIFICATION and STABILITY identification of the UNDEC status is trivial: under ST semantics arguments are either in the extension or attacked by the extension, but identifying the IN/OUT status is NP (JUSTIFICATION) and even in Π_2^p (STABILITY). On the other hand, PR-credulous-UNDEC-JUSTIFICATION is on a higher level in the polynomial hierarchy than PR-credulous-IN-JUSTIFICATION: while PR-credulous-IN-JUSTIFICATION is in NP (since verification of a positive instance can be done in polynomial time, given an admissible set containing the argument as a certificate), PR-credulous-UNDEC-JUSTIFICATION is Σ_2^p-hard as it can be reduced from 2-QBF.

For other semantics, the complexity of UNDEC-STABILITY does not differ from the IN-STABILITY complexity. This is for example the case for GR semantics and CP semantics for the sceptical justification status. In order to prove this, we next give a reduction from UNSAT (co-1-QBF), which is based on the reduction from [5, Definition 14] and illustrated in the black part of Figure 1.

Definition 7 (Reduction)**.** *Let (ϕ, X) be an instance of 1-QBF or co-1-QBF and let $\phi = \bigwedge_i c_i$, where $c_i = \bigvee_j \alpha_j$ for each clause c_i in ϕ and α_j are the literals over X that occur in clause c_i. We define the corresponding IAF for this instance as $\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset \rangle$, where:*

- $\mathcal{A} = \{x_i, \overline{x_i} \mid x_i \in X\} \cup \{\overline{c_i} \mid c_i \in \phi\} \cup \{\phi, \overline{\phi}\}$;
- $\mathcal{A}^? = \{g_i \mid x_i \in X\}$;

[2]Proofs of omitted results in the remainder of the paper can be found in the online appendix `https://www.uu.nl/onderzoek/ai-labs/nationaal-politielab-ai/stability-relevance-iafs`.

- $\mathcal{R} = \{(g_i, \overline{x_i}) \mid x_i \in X\} \cup \{(\overline{x_i}, x_i) \mid x_i \in X\} \cup \{(x_k, \overline{c_i}) \mid x_k \in c_i\} \cup \{(\overline{x_k}, \overline{c_i}) \mid \neg x_k \in c_i\} \cup \{(\overline{c_i}, \phi) \mid c_i \in \phi\} \cup \{(\phi, \overline{\phi}), (\overline{\phi}, \overline{\phi})\}$.

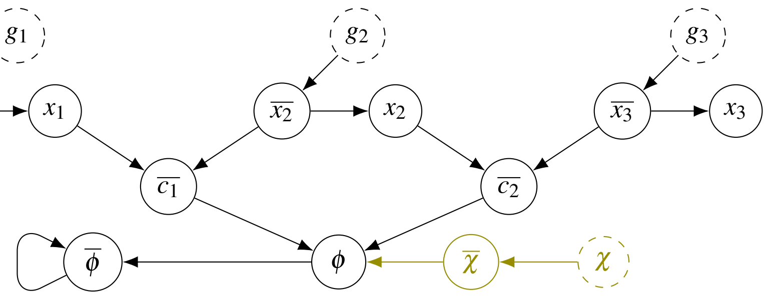

Figure 1. Visualisation of the IAF created for the clauses $c_1 = x_1 \vee \neg x_2$ and $c_2 = x_2 \vee \neg x_3$ using the reductions of Definition 7, only the black parts of the figure and Definition 11, also including the gold/gray part. We use this reduction for GR, CP and sceptical PR semantics.

The reduction is used in the proof of the following proposition.

Proposition 2. *GR-sceptical-UNDEC-STABILITY, GR-credulous-UNDEC-STABILITY and CP-credulous-UNDEC-STABILITY are CoNP-complete.*

Proof sketch. Let (ϕ, X) be a co-1-QBF (UNSAT) instance and let $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ be the IAF according to Definition 7. As GR semantics results in a single extension, which is the intersection of all CP extensions, the problems GR-sceptical-UNDEC-STABILITY, GR-credulous-UNDEC-STABILITY and CP-credulous-UNDEC-STABILITY coincide. The argument $\overline{\phi}$ can only be stable-UNDEC if for each specification of $\mathcal{I}$, the argument ϕ is OUT, which can only be the case if there is at least one clause in ϕ such that the corresponding argument for $\overline{c_i}$ is IN. Thus the following items are equivalent:

1. (ϕ, X) is a positive UNSAT instance;
2. the argument for $\overline{\phi}$ is stable-GR-sceptical-UNDEC in $\mathcal{I}$;
3. the argument for $\overline{\phi}$ is stable-GR-credulous-UNDEC in $\mathcal{I}$;
4. the argument for $\overline{\phi}$ is stable-CP-credulous-UNDEC in $\mathcal{I}$.

CoNP-hardness from UNDEC-STABILITY follows from CoNP-hardness of UNSAT. From Proposition 1 and the fact that the corresponding JUSTIFICATION problems are in P, we conclude CoNP-completeness. □

In the following proposition, we consider the sceptical variants of the UNDEC-STABILITY under CP and PR semantics.

Proposition 3. *CP-sceptical-UNDEC-STABILITY and PR-sceptical-UNDEC-STABILITY are CoNP-complete.*

Proof sketch. CP-sceptical-UNDEC-STABILITY and PR-sceptical-UNDEC-STABILITY are in CoNP, since negative instances $(\mathcal{I}, A)$ can be verified in polynomial time given a certificate (AF', S) such that $\mathcal{I}' \in F(\mathcal{I})$, AF' is the certain projection of $\mathcal{I}'$, $A \in \mathcal{A}$ and S is an admissible set of AF' containing either A itself or an argument attacking A. For hardness, note that the corresponding (CoNP-hard) JUSTIFICATION problems can be reduced to these STABILITY problems, leaving the uncertain part empty. □

Proposition 4 states that the PR-credulous-UNDEC-STABILITY problem is on the third level of the polynomial hierarchy. For the proof, we refer to the appendix.

Proposition 4. PR-*credulous*-UNDEC-STABILITY *is* Π_3^p*-complete.*

Finally, we consider UNDEC-STABILITY under ST semantics. Recall from Definition 1 that for each ST extension S of an AF, each argument in AF is either in S or attacked by some argument in S. Consequently, an argument can only be stable-ST-sceptical-UNDEC if the AF has no ST extension. We use this property in the following proposition:

Proposition 5. ST-*sceptical*-UNDEC-STABILITY *is CoNP-complete.*

Proof sketch. The problem is in CoNP, as a no-instance $(\mathcal{I}, A)$ can be verified in polynomial time given a certificate (AF', S) such that $\mathcal{I}' \in F(\mathcal{I})$, AF' is the certain projection of $\mathcal{I}'$ and S is a ST extension of AF'. If S is a ST extension then each argument in $\mathcal{A}$ is either in S or attacked by S; therefore no argument can be stable-ST-sceptical-UNDEC w.r.t. $\mathcal{I}$. For hardness, we can reduce from the CoNP-complete problem ST-sceptical-UNDEC-JUSTIFICATION. □

This result for sceptical acceptance may feel counterintuitive. Therefore, we consider an alternative version of justification and stability (based on [14]) for ST semantics, in which it is assumed that a ST extension exists.

Definition 8 (Sceptical-existent justification)**.** *Given an argumentation framework* $AF = \langle \mathcal{A}, \mathcal{R} \rangle$*, argument* $A \in \mathcal{A}$ *and label* LAB $\in \{$IN, OUT, UNDEC$\}$*, A is* ST-*sceptical-existent*-LAB *w.r.t. AF iff AF has at least one* ST *extension and A is* ST-*sceptical*-LAB *in AF.*

As no AF has a ST extension S in which some argument is neither in S, nor attacked by any argument in S, ST-sceptical-existent-UNDEC-STABILITY is False for all inputs. The same holds for ST-credulous-UNDEC-STABILITY, so these problems are trivial:

Proposition 6. ST-*credulous*-UNDEC-STABILITY *and* ST-*sceptical-existent*-UNDEC-STABILITY *are trivial.*

4. Relevance

For IAFs in which a given argument is not stable, a natural follow-up question would be which uncertainties should be resolved in order to reach a point where the argument is stable: these uncertainties are *relevant* to investigate in the given IAF. In this section, we will define the problem of relevance and study its complexity. First, we give some intuition on the notion of relevance in the context of stability in the following example.

Example 2. *Let* $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ *be the IAF illustrated in Figure 2, where only arguments A and C are certain and let j be* GR-*sceptical*-IN*; suppose that we want to know if argument A is j-stable. A is not j-stable in* $\mathcal{I}$*, but it will become j-stable in each* $\mathcal{I}' \in F(\mathcal{I})$ *such that* $\mathcal{I}' = \langle \mathcal{A}', \mathcal{A}^{?\prime}, \mathcal{R}', \mathcal{R}^{?\prime} \rangle$ *and* $D \notin \mathcal{A}' \cup \mathcal{A}^{?\prime}$ *or* $E \in \mathcal{A}'$*. Therefore, it would be relevant to investigate if D can be removed from the uncertain arguments and/or if E can be added to the certain arguments. Note that investigation of B does not contribute towards a stable situation and therefore would not be relevant.*

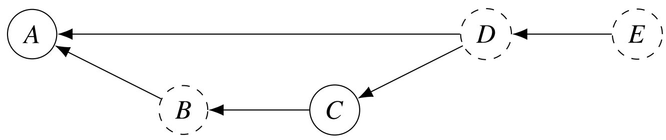

Figure 2. Visualisation of IAF $\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ used to illustrate the definition of relevance. In this figure, certain arguments in $\mathcal{A}$ are depicted as nodes with solid borders (i.e. A and C), while uncertain arguments in $\mathcal{A}^?$ have dashed borders (i.e. B, D and E). The arrows between them correspond to attacks in $\mathcal{R}$.

Before proceeding to a formal definition of relevance that matches the intuitions in the example above in Definition 10, we define the notion of minimal stable specifications.

Definition 9 (Minimal stable-j specification)**.** *Given an IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a certain argument $A \in \mathcal{A}$ and a justification status j, a minimal stable-j specification for A w.r.t. $\mathcal{I}$ is a specification $\mathcal{I}'$ in $F(\mathcal{I})$ such that A is stable-j in $\mathcal{I}'$ and there is no specification $\mathcal{I}''$ in $F(\mathcal{I})$ such that A is stable-j in $\mathcal{I}''$, $\mathcal{I}'' \neq \mathcal{I}'$ and $\mathcal{I}' \in F(\mathcal{I}'')$.*

Intuitively, the minimal stable-j specification for A is a specification in which A is stable-j, while A would not be stable-j in any specification with more uncertain elements.

Example 3. *Reconsider the IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ from Example 2:*

- *There are two minimal stable-*GR*-sceptical-*IN *specifications for A w.r.t. $\mathcal{I}$. These are $\langle \{A,C,E\}, \{B,D\}, \mathcal{R}, \mathcal{R}^? \rangle$ and $\langle \{A,C\}, \{B,E\}, \mathcal{R}, \mathcal{R}^? \rangle$;*
- *There are three specifications for A w.r.t. $\mathcal{I}$ for which A becomes stable-*GR*-sceptical-*OUT*: $\mathcal{I}_1 = \langle \{A,C,D\}, \{B\}, \mathcal{R}, \mathcal{R}^? \rangle$, $\mathcal{I}_2 = \langle \{A,C,D\}, \emptyset, \mathcal{R}, \mathcal{R}^? \rangle$ and $\mathcal{I}_3 = \langle \{A,B,C,D\}, \emptyset, \mathcal{R}, \mathcal{R}^? \rangle$. Only $\mathcal{I}_1$ is minimal, because both $\mathcal{I}_2$ and $\mathcal{I}_3$ are in $F(\mathcal{I}_1)$;*
- *None of the specifications is minimal stable-*GR*-sceptical-*UNDEC.

Using the notion of minimal stable-j specifications, we can now define j-RELEVANCE.

Definition 10 (j-RELEVANCE)**.** *Given an IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, an argument $A \in \mathcal{A}$, an uncertain argument or attack $U \in \mathcal{A}^? \cup \mathcal{R}^?$ and a justification status j,*

- *Addition of U is j-relevant for A w.r.t. $\mathcal{I}$ iff there is a minimal stable-j specification $\mathcal{I}' = \langle \mathcal{A}', \mathcal{A}^{?\prime}, \mathcal{R}', \mathcal{R}^{?\prime} \rangle$ for A w.r.t. $\mathcal{I}$ such that $U \in \mathcal{A}' \cup \mathcal{R}'$; and*
- *Removal of U is j-relevant for A w.r.t. $\mathcal{I}$ iff there is a minimal stable-j specification $\mathcal{I}' = \langle \mathcal{A}', \mathcal{A}^{?\prime}, \mathcal{R}', \mathcal{R}^{?\prime} \rangle$ for A w.r.t. $\mathcal{I}$ such that $U \notin \mathcal{A}' \cup \mathcal{A}^{?\prime} \cup \mathcal{R}' \cup \mathcal{R}^{?\prime}$.*

In other words, addition of an uncertain element U is j-relevant if a minimal stable-j specification can be reached by moving U from the certain to the uncertain part of the IAF $\mathcal{I}$; and removal of U is j-relevant if completely removing U from $\mathcal{I}$, possibly in combination with other actions, leads to a minimal stable-j specification.

Example 4. *Recall from Example 3 that only $\mathcal{I}_1 = \langle \{A,C,D\}, \{B\}, \mathcal{R}, \mathcal{R}^? \rangle$ is a minimal stable-*GR*-sceptical-*OUT *specification for A w.r.t. $\mathcal{I}$. Therefore, only addition of D and removal of E are* GR*-sceptical-*OUT*-relevant for A w.r.t. $\mathcal{I}$.*

Like for justification and stability status, we formulate the identification of j-RELEVANCE as a decision problem:

j-RELEVANCE of action **a**	
Given:	An incomplete argumentation framework $\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a justification status j, an action $\mathbf{a} \in \{addition, removal\}$, an argument $A \in \mathcal{A}$ and an uncertain argument or attack $U \in \mathcal{A}^? \cup \mathcal{R}^?$.
Question:	Is **a** of U j-relevant for A w.r.t. $\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$?

The following lemma, proven in the appendix, shows that the relevance of adding or removing an uncertain element can be validated by checking the justification status of the certain projections of two particular future specifications. This property will be useful for proving an upper bound on j-RELEVANCE.

Lemma 2. *Given an IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a certain argument $A \in \mathcal{A}$ and a justification status j:*

1. *For each $U \in \mathcal{A}^?$, addition of U is j-relevant for A w.r.t. $\mathcal{I}$ iff there exists some $\mathcal{I}' = \langle \mathcal{A}', \{U\}, \mathcal{R}', \emptyset \rangle \in F(\mathcal{I})$ such that A is **not** j in the certain projection of $\mathcal{I}'$, while A is j in the certain projection of $\langle \mathcal{A}' \cup \{U\}, \emptyset, \mathcal{R}', \emptyset \rangle$.*
2. *For each $U \in \mathcal{R}^?$, addition of U is j-relevant for A w.r.t. $\mathcal{I}$ iff there exists some $\mathcal{I}' = \langle \mathcal{A}', \emptyset, \mathcal{R}', \{U\} \rangle \in F(\mathcal{I})$ such that A is **not** j in the certain projection of $\mathcal{I}'$, while A is j in the certain projection of $\langle \mathcal{A}', \emptyset, \mathcal{R}' \cup \{U\}, \emptyset \rangle$.*
3. *For each $U \in \mathcal{A}^?$, removal of U is j-relevant for A w.r.t. $\mathcal{I}$ iff there exists some $\mathcal{I}' = \langle \mathcal{A}', \{U\}, \mathcal{R}', \emptyset \rangle \in F(\mathcal{I})$ such that A is j in the certain projection of $\mathcal{I}'$, while A is **not** j in the certain projection of $\langle \mathcal{A}' \cup \{U\}, \emptyset, \mathcal{R}', \emptyset \rangle$.*
4. *For each $U \in \mathcal{R}^?$, removal of U is j-relevant for A w.r.t. $\mathcal{I}$ iff there exists some $\mathcal{I}' = \langle \mathcal{A}', \{U\}, \mathcal{R}', \emptyset \rangle \in F(\mathcal{I})$ such that A is j in the certain projection of $\mathcal{I}'$, while A is **not** j in the certain projection of $\langle \mathcal{A}', \emptyset, \mathcal{R}' \cup \{U\}, \emptyset \rangle$.*

In the following proposition, we use the results from Lemma 2 to prove a general upper bound on the complexity of j-RELEVANCE.

Proposition 7 (Upper bound j-RELEVANCE)**.** *Given an IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a certain argument $A \in \mathcal{A}$, an uncertain argument or attack $U \in \mathcal{A}^? \cup \mathcal{R}^?$ and a justification status j, if the complexity of deciding j's justification status in $\langle \mathcal{A}, \mathcal{R} \rangle$ is $\mathcal{C}$, then an upper bound on the problem of deciding if addition and/or removal of U is j-relevant for A w.r.t. $\mathcal{I}$ is $NP^{\mathcal{C}}$.*

In order to prove a lower bound of j-RELEVANCE for GR and CP sceptical semantics, we use the following reduction. This reduction is illustrated in Figure 1 in gold.

Definition 11 (Reduction relevance)**.** *Let (ϕ, X) be an instance of 1-QBF or co-1-QBF let $\phi = \bigwedge_i c_i$ and $c_i = \bigvee_j \alpha_j$ for each clause c_i in ϕ, where α_j are the literals over X that occur in clause c_i. Let $\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset \rangle$ be the reduction from Definition 7 of this instance. We define the corresponding IAF for this instance as $\langle \mathcal{A}', \mathcal{A}^{?\prime}, \mathcal{R}', \emptyset \rangle$, where $\mathcal{A}' = \mathcal{A} \cup \{\overline{\chi}\}$; $\mathcal{A}^{?\prime} = \mathcal{A}^? \cup \{\chi\}$; and $\mathcal{R}' = \mathcal{R} \cup \{(\chi, \overline{\chi}), (\overline{\chi}, \phi)\}$.*

Using the reduction above, we can now give tight complexity bounds for GR and CP semantics for the sceptical justification status.

Proposition 8. GR-*sceptical*-IN-RELEVANCE, GR-*credulous*-IN-RELEVANCE *and* CP-*sceptical*-IN-RELEVANCE *are NP-complete.*

Proof sketch. Given an instance of 1-QBF (SAT), let $\mathcal{I}$ be the IAF constructed according to Definition 11 where $Y = \emptyset$. Addition of χ is GR-sceptical-IN-relevant for ϕ w.r.t. $\mathcal{I}$ iff there is some $\mathcal{I}' \in F(\mathcal{I})$ such that ϕ is GR-sceptical-IN, having χ in its certain arguments iff the SAT instance is True. NP-completeness follows from Proposition 8 and the fact that the corresponding JUSTIFICATION problems are in P. $\square$

5. Related work

The computational complexity of various problems defined on argumentation frameworks is well-studied; see [15] for an overview. Most studies only identify IN arguments and do not distinguish other justification statuses; notable exceptions are [16] and [17], but neither of these works give complexity results for separate statuses, as we do.

Complexity studies on problems defined on IAFs emerged more recently. For example, variants of the verification problem on IAFs are studied in [4]. The problems of stability and relevance differ from the verification problem as they are defined on arguments rather than sets of arguments. More related is [5]: the authors study potential and necessary credulous and sceptical acceptance in IAFs, where necessary sceptical acceptance of a given argument A, for example, means that in each specification's certain projection, each extension (under a given semantics) contains A. The notions of necessary credulous and sceptical acceptance are very similar to specific stability problems: in fact, we used results regarding their complexity for proving the complexity of stable-IN statuses. Finally, the notion of stability, which was originally defined on structured argumentation frameworks in [7], is transposed to the context of IAFs in [6] and preliminary complexity results for stability under four semantics are provided. In our work, we define a more fine-grained notion of stability and provide more precise complexity characterisations.

Our notion of relevance has not been introduced or studied before. It is related to the notion of influenced sets in e.g. [18], which intuitively are sets of arguments whose justification status may change after an update. This notion is however less strict than relevance: there are situations in which some argument A would be in the influenced set of adding an uncertain attack (B,C), while addition of (B,C) is not relevant for A.

6. Conclusion

We studied the complexity of detecting stability and relevance in incomplete argumentation frameworks. First, we redefined stability [7,8,6] on IAFs. Our definition is a more fine-grained notion than the existing definition on IAFs [6], since it distinguishes between IN, OUT and UNDEC justification statuses. This distinction is appropriate in for example applications in inquiry [7,8], where a dialogue discussing a given argument should be terminated if more information cannot change the argument's (exact) justification status.

As second main contribution of this paper, we introduced the notion of relevance, which has not been studied before in the context of stability, and analysed its complexity. Returning to the application in inquiry, the identification of relevant elements can be used to select the next question, reaching a stable situation in an efficient way.

It is however unlikely that the stability and relevance problem itself can be solved efficiently for all inputs: our complexity analysis revealed that the nontrivial variants of the relevance and stability problems have a complexity ranging from the first to the third level of the polynomial hierarchy; see Table 1 for an overview. Interestingly, even within the same semantics, there are differences in the complexity of UNDEC-STABILITY problems and the corresponding IN-STABILITY problems – we consider this to be an additional reason to study a fine-grained notion of stability and relevance.

In future work we will complete Table 1, by studying the computational complexity of relevance for the other semantics and UNDEC status. In addition, to apply these theoretical concepts in practice, we plan to develop algorithms for evaluating or estimating stability and relevance. Finally, we will study stability and relevance in structured argumentation frameworks, such as a dynamic version of ASPIC^{+}, for various semantics.

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